ON MONOIDAL EQUIVALENCES AND ANN-EQUIVALENCES

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1 Introduction

The equivalence between a monoidal category and a strict one has been proved by some authors such as Nguyen Duy Thuan [8], Christian Kassel [2], Peter Schauenburg [7]. In this paper, we show another proof of the problem by constructing a strict monoidal category $M(\mathcal{C})$ consisting of M-functors and M-morphisms of a category $\mathcal{C}$ and we prove $\mathcal{C}$ is equivalent to it. The proof is based on a basic character of monoidal equivalences. Ideas and techniques of these proofs have been used to prove the equivalence between an Ann-category and an almost strict Ann-category [5].
The basic concepts of monoidal categories are shown in [2],[3].

2 Monoidal equivalences

A monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$ is a category $\mathcal{C}$ which is equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$; with an object I which is called *the unit* of monoidal category and with following natural isomorphisms

$$a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$
$$l_A : I \otimes A \to A$$
$$r_A : A \otimes I \to A$$

which are called respectively *the associativity constraint, the left unit constraint* and *the right unit constraint*. These constraints have to satisfy the Pentagon

Axiom

$$(a_{A,B,C} \otimes id_D)\ a_{A,B\otimes C,D}\ (id_A \otimes a_{B,C,D}) = a_{A\otimes B,C,D}\ a_{A,B,C\otimes D},$$

and the Triangle Axiom

$$id_A \otimes l_B = (r_A \otimes id_B) a_{A,I,B}.$$

A monoidal category is said to be *strict* if the associativity and unit constraints $a,\ l,\ r$ are all identities of the category.

Let $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ and $\mathcal{D} = (\mathcal{D}, \otimes, I', a', l', r')$ be monoidal categories. *A monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triplet $(F, \widetilde{F}, \hat{F})$ where $F : \mathcal{C} \to \mathcal{D}$ is a functor, $\hat{F}$ is an isomorphism from I' to FI, and a natural isomorphism

$$\widetilde{F}_{A,B} : FA \otimes FB \to F(A \otimes B)$$

making diagrams

$$\begin{array}{ccccc}
FA\otimes(FB\otimes FC) & \xrightarrow{\mathrm{id}\otimes\widetilde{F}} & FA\otimes F(B\otimes C) & \xrightarrow{\widetilde{F}} & F(A\otimes(B\otimes C)) \\
{\scriptstyle a'}\downarrow & & & & \downarrow{\scriptstyle F(a)} \\
(FA\otimes FB)\otimes FC & \xrightarrow{\widetilde{F}\otimes\mathrm{id}} & F(A\otimes B)\otimes FC & \xrightarrow{\widetilde{F}} & F((A\otimes B)\otimes C)
\end{array}$$

$$\begin{array}{ccc}
FA\otimes FI & \xrightarrow{\widetilde{F}} & F(A\otimes I) \\
{\scriptstyle id\otimes\hat{F}}\uparrow & & \downarrow{\scriptstyle F(l)} \\
FA\otimes I' & \xrightarrow{r'} & FA
\end{array}
\qquad
\begin{array}{ccc}
FI\otimes FA & \xrightarrow{\widetilde{F}} & F(I\otimes A) \\
{\scriptstyle \hat{F}\otimes id}\uparrow & & \downarrow{\scriptstyle F(r)} \\
I'\otimes FA & \xrightarrow{l'} & FA
\end{array}$$

commute for all objects $A,\ B,\ C$ in $\mathcal{C}$.

If $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories, a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is *an monoidal equivalence* if there exists a monoidal functor $G : \mathcal{D} \to \mathcal{C}$ such that there exist monoidal natural isomorphisms $\alpha : G.F \to \mathrm{id}_{\mathcal{C}}$ and $\beta : F.G \to \mathrm{id}_{\mathcal{D}}$.

We say that $\mathcal{C}$ and $\mathcal{D}$ are *monoidal equivalent* if there exists a monoidal equivalence between them.
In this paper, we use the term *a tensor category* to refer to a category equiped with a tensor product.
The first main result of this paper is the following theorem.

Theorem 1. *Any monoidal category is monoidal equivalent to a strict one.*

To prove the theorem, for any given monoidal category $\mathcal{C}$, we construct a strict one $M(\mathcal{C})$ which is monoidal equivalent to $\mathcal{C}$.
We first point out a simple character of monoidal equivalences.

Theorem 2. *A monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is a monoidal equivalence if and only if F is an equivalence of categories.*

Proof. Assume $F:\mathcal{C}\to\mathcal{D}$ be a monoidal functor with a natural isomorphism

$$\widetilde{F}_{X,Y}:FX\otimes FY\to F(X\otimes Y).$$

Since F is an equivalence of categories, there exists a functor $G:\mathcal{D}\to\mathcal{C}$ and natural transformations $\alpha:GF\to \mathrm{id}_{\mathcal{C}}$, $\beta:FG\to \mathrm{id}_{\mathcal{D}}$. Moreover, we can choose $\alpha,\ \beta$ satisfying $F(\alpha_A)=\beta_{FA}\ ;\ G(\beta_B)=\alpha_{GB}$ for all objects $A\in\mathcal{C},\ B\in\mathcal{D}$. The natural isomorphism

$$\widetilde{G}_{U,V}:GU\otimes GV\to G(U\otimes V)$$

for $U,V\in\mathcal{D}$ is defined by the outside region in the following commutative diagram

$$\begin{tikzcd}
 & FG(UV) \arrow[r, "\beta_{UV}"] & UV \arrow[dd, "id"] \\
F(GU.GV) \arrow[ur, "F(\widetilde{G})"] \arrow[r, phantom, "(1)"] & {} \arrow[r, phantom, "(2)"] & {} \\
 & FGU.FGV \arrow[ul, "\widehat{F}"] \arrow[uu, "\widetilde{FG}"'] \arrow[r, "\beta_U\otimes\beta_V"'] & UV
\end{tikzcd}$$

In this diagram, region (1) commutes by the definition of $\widehat{FG}$. It follows that region (2) commutes then β is $\otimes$-transformation. Finally, α is a $\otimes$-transformation thanks to the naturalism of β. ☐

We now construct the strict monoidal category $M(\mathcal{C})$ of a monoidal one $(\mathcal{C},\otimes,a,(I,l,r))$ as follows:

Definition. *A M-functor of monoidal category $\mathcal{C}$ is a pair $(F,\overline{F})$ consisting of a functor $F:\mathcal{C}\to\mathcal{C}$ and a natural isomorphism*

$$\overline{F}_{A,B}:F(A\otimes B)\to FA\otimes B,$$

making the following diagrams

$$\begin{tikzcd}
F(A\otimes(B\otimes C)) \arrow[rr, "F(a)"] \arrow[d, "\overline{F}"'] & & F((A\otimes B)\otimes C) \arrow[d, "\overline{F}"] \\
FA\otimes(B\otimes C) \arrow[r, "a"] & (FA\otimes B)\otimes C & F(A\otimes B)\otimes C \arrow[l, "\overline{F}\otimes id_C"']
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
F(A\otimes I) \arrow[rr, "\overline{F}"] \arrow[dr, "\overline{F}(r_A)"'] & & FA\otimes I \arrow[dl, "r_{FA}"] \\
 & FA &
\end{tikzcd} \tag{2}$$

commute.

Definition. *Let* $(F, \overline{F})$ *and* $(G, \overline{G})$ *be M-functors. A M-transformation* $\alpha : (F, \overline{F}) \to (G, \overline{G})$ *is a natural transformation* $\alpha : F \to G$ *making the following diagram*

$$\begin{tikzcd} F(A \otimes B) \arrow[r, "\overline{F}"] \arrow[d, "\alpha_{A\otimes B}"'] & FA \otimes B \arrow[d, "\alpha_A \otimes id_B"] \\ G(A \otimes B) \arrow[r, "\overline{G}"] & GA \otimes B \end{tikzcd} \tag{3}$$

commute.

The composition of two M-transformations is known as the composition of usual natural transformations. Readers can easily verify that the composition of M-transformations is also a M-transformation.

Example. For any object X of $\mathcal{C}$, the pair $(L^X, \bar{L}^X)$ defined as $L^X(A) = X \otimes A$, $L^X(u) = id_X \otimes u$, $\overline{L}^X_{A,B} = a_{X,A,B}$ is a M-functor of $\mathcal{C}$.
For any pair (X, Y) of objects of $\mathcal{C}$ and a morphism $f : X \to Y$, the natural transformation $\alpha : L^X \to L^Y$ given by $\alpha_A = f \otimes A$ is a M-transformation of $\mathcal{C}$.

For any monoidal category $\mathcal{C}$, by $M(\mathcal{C})$ we denote the category whose objects are M-functors and whose morphisms are M-transformations of $\mathcal{C}$. We now equip $M(\mathcal{C})$ with the structure of a strict tensor category.

Lemma 3. *Let* $(F, \overline{F})$ *and* $(G, \overline{G})$ *be M-functors of* $\mathcal{C}$*. Then, the composition* FG *is also a M-functor with the natural isomorphism* $\overline{FG}$ *defined by following commutative diagram*

$$\begin{tikzcd} FG(A \otimes B) \arrow[rr, "\overline{FG}"] \arrow[dr, "F(\overline{G})"'] & & FGA \otimes B \\ & F(GA \otimes B) \arrow[ur, "\overline{F}"'] & \end{tikzcd} \tag{4}$$

for any pair (A, B) *of objects of* $\mathcal{C}$*.*

Proof. Consider the following diagram

$$
\begin{tikzcd}
\mathrm{FG((A\otimes B)\otimes C)} \arrow[d, "F(\overline{G})"] \arrow[dd, bend right=60, "\overline{FG}"'] \arrow[rrd, phantom, "(1)"] & & \mathrm{FG(A\otimes(B\otimes C))} \arrow[ll, "\mathrm{FG(a)}"'] \arrow[d, "F(\overline{G})"] \arrow[dd, bend left=60, "\overline{FG}"] \\
\mathrm{F(G(A\otimes B)\otimes C)} \arrow[d, "\overline{F}"] \arrow[r, phantom, "(2)"] & \mathrm{F((GA\otimes B)\otimes C)} \arrow[l, "F(\overline{G}\otimes id)"'] \arrow[d, "\overline{F}"] \arrow[r, phantom, "(4)"] & \mathrm{F(GA\otimes(B\otimes C))} \arrow[l, "\mathrm{F(a)}"'] \arrow[d, "\overline{F}"] \\
\mathrm{FG(A\otimes B)\otimes C} \arrow[dr, "\overline{FG}\otimes id"'] \arrow[r, phantom, "(3)"] & \mathrm{F(GA\otimes B)\otimes C} \arrow[l, "F(\overline{G})\otimes id"'] \arrow[d, "\overline{F}\otimes id"] & \mathrm{FGA\otimes(B\otimes C)} \arrow[dl, "\mathrm{a}"] \\
& \mathrm{(FGA\otimes B)\otimes C} &
\end{tikzcd}
$$

(5) (6)

In the above diagram, regions (1) and (4) commute by the definition of M-functors, (2) one commutes thanks to the naturality of $\overline{F}$, (3), (5) and (6) ones commute due to the denifition of $\overline{FG}$. Then the outside region which is the diagram (1) for the pair $(FG, \overline{FG})$ commutes.

Now we consider the following diagram

$$
\begin{tikzcd}
FG(A\otimes I) \arrow[rr, "\overline{FG}"] \arrow[dr, "F(\overline{G})"] \arrow[ddr, bend right, "FG(r)"'] & (1) & FGA\otimes I \arrow[ddl, bend left, "r_{FGA}"] \\
& {}^{(2)}F(GA\otimes I)^{(3)} \arrow[ur, "\overline{F}"] \arrow[d, "F(r)"] & \\
& FGA &
\end{tikzcd}
$$

In the above diagram, region (1) commutes owing to the definition of $\overline{FG}$, (2), (3) ones commute because of the definition of M-functors. This follows that the outside region which is the diagram (2) for the pair $(FG, \overline{FG})$ commutes. $\square$

Lemma 4. *For any pair* $((F,\overline{F}) \xrightarrow{\alpha} (F',\overline{F}'); (G,\overline{G}) \xrightarrow{\beta} (G',\overline{G}'))$ *of M-transformations of* $\mathcal{C}$*, the natural transformation*

$$\alpha * \beta : FG \to F'G'$$

making regions (1) and (2) of the following diagram

$$
\begin{tikzcd}
 & FG'A \arrow[dr, "\alpha_{G'A}"] \arrow[d, phantom, "(1)"] & \\
FGA \arrow[ur, "F(\beta_A)"] \arrow[rr, dotted, "(\alpha*\beta)_A"] \arrow[dr, "\alpha_{GA}"'] & {} \arrow[d, phantom, "(2)"] & F'G'A \\
 & F'GA \arrow[ur, "F'(\beta_A)"'] &
\end{tikzcd}
\tag{5}
$$

commute is a M-transformation from $(FG, \overline{FG})$ *to* $(F'G', \overline{F'G'})$.

Proof. In the following diagram, regions (1), (6) commute thanks to definitions of $\overline{FG}$ and $\overline{F'G'}$, the (3), (5) ones commute by the naturalism of $\alpha, \overline{F}'$, the (2),(4) ones commute due to the definition of M-transformations α, β, the (7), (8) ones commute owing to the definition of $\alpha * \beta$. Hence the outside region that is the diagram (3) for $\alpha * \beta$ commutes.

$$
\begin{tikzcd}
FG(A\otimes B) \arrow[rr, "\overline{FG}"] \arrow[dr, "F(\overline{G})"] \arrow[dd, "\alpha_{G(A\otimes B)}"'] \arrow[dddd, bend right=60, "(\alpha*\beta)_{A\otimes B}"'] & {} \arrow[d, phantom, "(1)"] & FGA\otimes B \arrow[dd, "\alpha_{GA}\otimes id"] \arrow[dddd, bend left=60, "(\alpha*\beta)_{A\otimes B}"] \\
{} \arrow[r, phantom, "(2)"] & F(GA\otimes B) \arrow[ur, "\overline{F}"'] \arrow[d, "\alpha_{GA\otimes B}"'] \arrow[r, phantom, "(3)"] & {} \\
F'G(A\otimes B) \arrow[r, "F'(\overline{G})"] \arrow[dd, "F'(\beta)"'] \arrow[dd, phantom, bend right=40, "(7)"] & F'(GA\otimes B) \arrow[r, "F'"] \arrow[d, "F'(\beta\otimes id)"'] & F'GA\otimes B \arrow[dd, "F'(\beta)\otimes id"] \arrow[dd, phantom, bend left=40, "(8)"] \\
{} \arrow[r, phantom, "(5)"] & F'(G'A\otimes B) \arrow[dr, "F'"] \arrow[d, phantom, "(6)"] \arrow[r, phantom, "(4)"] & {} \\
F'G'(A\otimes B) \arrow[ur, "F'(\overline{G'})"] \arrow[rr, "\overline{F'G'}"'] & {} & F'G'A\otimes B
\end{tikzcd}
\tag{6}
$$

□

Proposition 5. *We can equip* $M(\mathcal{C})$ *with a structure of tensor category by the tensor product defined as follows*

$$(F, \overline{F}) \oplus (G, \overline{G}) = (FG, \overline{FG})$$
$$\alpha \oplus \beta = \alpha * \beta : FG \to F'G'$$

Proof. It is easy for readers to check that the above tensor product is well-defined. □

Now we show the associativity and unit constraints of the tensor category $M(\mathcal{C})$.
By the associative property of the composition of functors we clearly see that the identity is the associativity contraint with the tensor product $\oplus$ of $M(\mathcal{C})$. Moreover, we can choose the pair $I^* = (\mathrm{id}, \overline{\mathrm{id}})$ where id is the identity functor and $\overline{\mathrm{id}}$ is the identity natural isomorphism of the unit object.

Proposition 6. *The tensor category $M(\mathcal{C})$ is a strict monoidal category.*

In the rest of the section, we prove the Theorem 1 by showing that every monoidal category $\mathcal{C}$ is monoidal equivalent to the strict one $M(\mathcal{C})$ of it. We do it in the following steps:
Step 1: Define a monoidal functor $\Phi : C \to M(\mathcal{C})$ as follows:

$$\Phi(A) = (L^A, \overline{L}^A),$$
$$\Phi(f)_X = f \otimes id_X : A \otimes X \to B \otimes X,$$

for any objects A, X and any morphism $f : A \to B$ of $\mathcal{C}$. From the above example, $(L^A, \overline{L}^A)$ is a M-functor and $\Phi(f)$ is a M-transformation. Furthermore, it is easy to check that the triplet $(\Phi, \widetilde{\Phi}, \widehat{\Phi})$ where $\widetilde{\Phi}$ is a natural tensor isomorphism and $\widehat{\Phi}$ is an isomorphism defined as follows

$$\widetilde{\Phi}_{A,B} : \Phi(A) \oplus \Phi(B) \to \Phi(A \otimes B),$$
$$\widetilde{\Phi}_{A,B}(X) = a_{A,B,X} : A \otimes (B \otimes X) \to (A \otimes B) \otimes X$$

and

$$\widehat{\Phi} : I^* \to \Phi(I),$$
$$\widehat{\Phi}_X = l_X^{-1}$$

is a monoidal functor.
Step 2: In this step, we prove the triplet $(\Phi, \widetilde{\Phi}, \widehat{\Phi})$ is a monoidal equivalence. In order to do this we have to exhibit a functor which is the inverse eqivalence to Φ. Consider the functor

$$\Gamma : M(\mathcal{C}) \to \mathcal{C}$$
$$\Gamma(F, \overline{F}) = F(I),$$
$$\Gamma(\alpha) = \alpha_I : F(I) \to G(I),$$

for any M-functor $(F, \overline{F})$ and any M-transformation $\alpha : (F, \overline{F}) \to (G, \overline{G})$.
Observe that $\Gamma\Phi(f) = \Gamma(\Phi f) = (\Phi f)(I) = f \otimes id_I : A \otimes I \to B \otimes I$ for any morphism $f : A \to B$ of $\mathcal{C}$. Then we have the natural isomorphism $r : \Gamma\Phi \cong \mathrm{id}_{\mathcal{C}}$, where r is a right unit constraint of $\mathcal{C}$. We now prove that there exists an isomorphism ρ between $\Phi\Gamma$ and $id_{M(\mathcal{C})}$ of $M(\mathcal{C})$. We have

$$(\Phi\Gamma)(F, \overline{F}) = \Phi(f(I)) = (L^{FI}, \overline{L}^{FI})$$

$$\Phi\Gamma(\alpha) = \Phi(\alpha_I) : (L^{FI}, \overline{L}^{FI}) \to (L^{GI}, \overline{L}^{GI})$$

where $\alpha : (F, \overline{F}) \to (G, \overline{G})$.
As for any object $X \in \mathcal{C}$

$$L^{FI}(X) = FI \otimes X \xleftarrow{\overline{F}} F(I \otimes X) \xrightarrow{F(l_X)} FX,$$

we define the natural isomorphism $\rho : \Phi\Gamma \cong \mathrm{id}_{M(\mathcal{C})}$ as follows

$$\rho_{(F,\overline{F})} : (L^{FI}, \overline{L}^{FI}) \to (F, \overline{F}),$$
$$\rho_{(F,\overline{F})}(X) = F(l_X)(\overline{F}_{I,X})^{-1}.$$

Consider the following diagram

$$\begin{tikzcd}
FI \otimes (A \otimes X) & F(I \otimes (A \otimes X)) \arrow[l, "\overline{F}_{I,A\otimes X}"'] \arrow[r, "F(l_{A\otimes X})"] \arrow[d, "F(a)"'] \arrow[d, phantom, shift left=4ex, "(2)"] & F(A \otimes X) \arrow[dd, "\overline{F}"] \\
 & F((I \otimes A) \otimes X) \arrow[ur, "F(l_A \otimes X)"'] \arrow[d, "\overline{F}_{I\otimes A,X}"'] \arrow[dr, phantom, "(3)"] & \\
(FI \otimes A) \otimes X \arrow[uu, "\overline{L}^{FI}=a^{-1}"] & F(I \otimes A) \otimes X \arrow[l, "\overline{F}_{I,A\otimes X}"] \arrow[r, "F(l_A)\otimes X"'] & FA \otimes X
\end{tikzcd} \tag{7}$$

In the diagram, region (1) commutes because it is the diagram (1) for M-functor $(F, \overline{F})$, the (2) one commutes by the compatibility of the associativity constraint a with the unit constraint (I, l, r) (image through F), the (3) one commutes due to the naturalism of $\overline{F}$. Hence, the outside region which is the diagram (3) for M-transformation $\rho_{(F,\overline{F})}$ commutes. This follows that the definition of the natural isomorphism ρ is well-defined.
As Φ is an equivalence of categories and from Theorem 2 we derive Φ be a monoidal equivalence.

3 Ann-equivalences of Ann-categories

The concepts of Ann-categories and Ann-functors have first presented in [4]. It's been constructed based on the concepts of monoidal categories and symetric monoidal categories.
An *Ann-category* consists of

(i) A category $\mathcal{A}$ and two bifunctors $\oplus, \otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$;

(ii) A fixed object $O \in \mathcal{A}$ together with natural isomorphisms a^+, c, g, d such that $(\mathcal{A}, \oplus, a^+, c, (O, g, d))$ is a Pic-category;

(iii)A fixed object $I \in \mathcal{A}$ together with natural isomorphisms a, l, r such that $(\mathcal{A}, \otimes, a, (I, l, r))$ is a monoidal category;

(iv) The distributive natural isomorphisms (the left and right distributive constraints)

$$\mathfrak{L}_{A,X,Y} : A \otimes (X \oplus Y) \to (A \otimes X) \oplus (A \otimes Y);$$
$$\mathfrak{R}_{A,X,Y} : (X \oplus Y) \otimes A \to (X \otimes A) \oplus (Y \otimes A)$$

satisfy the axiomatics of Ann-categories (see [4],[6]).

Hereafter, for any objects A and B we denote AB instead of $A \otimes B$. However, for morphisms we still denote $f \otimes g$ to avoid confusion with a composition.

Definition. *Let $\mathcal{A}$, $\mathcal{B}$ be Ann-categories. An Ann-functor from $\mathcal{A}$ to $\mathcal{B}$ is a triplet $(F, \breve{F}, \widetilde{F})$ where $(F, \breve{F})$ is a $\oplus$-symetric monoidal functor and $(F, \widetilde{F})$ is a $\otimes$-monoidal functor making two following diagrams commute.*

$$\begin{tikzcd}
F(X(Y \oplus Z)) \arrow[r, "\widetilde{F}"] \arrow[d, "F(\mathfrak{L})"'] & FXF(Y \oplus Z) \arrow[r, "id \otimes \breve{F}"] & FX(FY \oplus FZ) \arrow[d, "\mathfrak{L}'"] \\
F(XY \oplus XZ) \arrow[r, "\breve{F}"] & F(XY) \oplus F(XZ) \arrow[r, "\hat{F} \oplus \widetilde{F}"] & FXFY \oplus FXFZ
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
F((X \oplus Y)Z) \arrow[r, "\widetilde{F}"] \arrow[d, "F(\mathfrak{R})"'] & F(X \oplus Y)FZ \arrow[r, "\breve{F} \otimes \mathrm{id}"] & (FX \oplus FY)FZ \arrow[d, "\mathfrak{R}'"] \\
F(XZ \oplus YZ) \arrow[r, "\breve{F}"] & F(XZ) \oplus F(YZ) \arrow[r, "\hat{F} \oplus \widetilde{F}"] & FXFZ \oplus FYFZ
\end{tikzcd} \tag{2}$$

Definition. *Let F, G be Ann-functors. A natural transformation $\varphi : F \to G$ is called an Ann-transformation if it is both a $\oplus$-transformation and a $\otimes$-transformation, i.e the following diagrams commute*

$$\begin{tikzcd}
F(A \oplus B) \arrow[r, "\breve{F}"] \arrow[d, "\varphi_{A \oplus B}"'] & FA \oplus FB \arrow[d, "\varphi_A \oplus \varphi_B"] \\
G(A \oplus B) \arrow[r, "\breve{G}"] & GA \oplus GB
\end{tikzcd} \qquad
\begin{tikzcd}
F(AB) \arrow[r, "\widetilde{F}"] \arrow[d, "\varphi_{AB}"'] & FAFB \arrow[d, "\varphi_A \otimes \varphi_B"] \\
G(AB) \arrow[r, "\widetilde{G}"] & GAGB
\end{tikzcd}$$

Definition. *An Ann-functor $F : \mathcal{A} \to \mathcal{B}$ is called an Ann-equivalence if there exists another one $G : \mathcal{B} \to \mathcal{A}$ and natural Ann-isomorphisms $\alpha : GF \cong \mathrm{id}_{\mathcal{A}},\ \beta : FG \cong \mathrm{id}_{\mathcal{B}}$.*

Two Ann-categories are Ann-equivalent if there exists an Ann-equivalence between them.

An Ann-category is called an almost strict one if its constraints are indentities except for one distributivity constraint (left or right) and commutativity constraint.

In this section, the main result is the following theorem

Theorem 7. *Any Ann-category is Ann-equivalent to an almost strict one.*

This result have been presented (in Vietnamese) in [5]. In the paper, we do again by a more complete way.

Theorem 8. *An Ann-functor $F : \mathcal{A} \to \mathcal{B}$ is an Ann-equivalence iff F is an equivalence of categories.*

In order to prove this theorem we first prove some lemmas:

Lemma 9. *Let $\alpha : F \cong G$ be both a natural $\oplus$-transformation and a natural $\otimes$-one. Then, if functor $F : \mathcal{A} \to \mathcal{B}$ is compatible to the left distributive constraints $\mathfrak{L}$, $\mathfrak{L}'$ then functor G is also compatible to $\mathfrak{L}$, $\mathfrak{L}'$. Similarly for the right distributive constraints $\mathfrak{R}$, $\mathfrak{R}'$.*

Proof. Consider the following diagram

$$\begin{tikzcd}
G(X(Y\oplus Z)) \arrow[r, "\widetilde{G}"] \arrow[ddd, bend right=60, "G(\mathfrak{L})"'] \arrow[dr, phantom, "(1)"] & GXG(Y\oplus Z) \arrow[r, "GX\otimes \breve{G}"] \arrow[dr, phantom, "(2)"] & GX(GY\oplus GZ) \arrow[ddd, bend left=60, "\mathfrak{L}'"] \\
F(X(Y\oplus Z)) \arrow[u, "\alpha"] \arrow[r, "\widetilde{F}"] \arrow[d, "F(\mathfrak{L})"] \arrow[d, phantom, shift right=12, "(6)"] \arrow[dr, phantom, "(3)"] & FXF(Y\oplus Z)) \arrow[u, "\alpha\otimes\alpha"'] \arrow[r, "FX\otimes \breve{F}"] & FX(FY\oplus FZ) \arrow[u, "\alpha\otimes(\alpha\oplus\alpha)"'] \arrow[d, "\mathfrak{L}'"'] \arrow[d, phantom, shift left=12, "(7)"] \\
F(XY\oplus XZ) \arrow[r, "\breve{F}"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(4)"] & F(XY)\oplus F(XZ) \arrow[r, "\widetilde{F}\oplus\widetilde{F}"] \arrow[d, "\alpha\oplus\alpha"'] \arrow[dr, phantom, "(5)"] & FXFY\oplus FXFZ \arrow[d, "(\alpha\otimes\alpha)\oplus(\alpha\otimes\alpha)"'] \\
G(XY\oplus XZ) \arrow[r, "\breve{G}"] & G(XY)\oplus G(XZ) \arrow[r, "\widetilde{G}\oplus\widetilde{G}"] & GXGY\oplus GXGZ
\end{tikzcd}$$

In the diagram, regions (1), (5) commute since α is a natural $\otimes$-transformation; the (2), (4) ones commute as α is a natural $\oplus$-transformation; the (3) one commutes thanks to the compatibility of G to $\mathfrak{L}$, $\mathfrak{L}'$; the (6), (7) ones commute by the naturalism of α and $\mathfrak{L}'$. Hence, the outside commutes, it implies that G is compatible to $\mathfrak{L}$, $\mathfrak{L}'$. $\square$

From the above lemma, we prove the following lemma:

Lemma 10. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors. If F is compatible to the left distributive constraints $\mathfrak{L}$, $\mathfrak{L}'$ and there exists a natural isomorphism $\alpha : FG \cong \mathrm{id}_{\mathcal{B}}$ which is both a natural $\oplus$-isomorphism and a $\otimes$-one then G is also compatible to $\mathfrak{L}$, $\mathfrak{L}'$.*

Proof. We will show that functor F satisfies the commutative diagram (1). In the following diagram, regions (1), (5), (2), (7) commute by the definition of $\breve{FG}$ and $\widehat{FG}$; the (3), (6) ones commute thanks to the naturality of $\widehat{F}$ and $\breve{F}$; the (8) one commutes owing to the compatibility of F to $\mathfrak{L}$, $\mathfrak{L}'$. Applying Lemma 9 we have the compatibility of FG to $\mathfrak{L}$, $\mathfrak{L}'$, therefore the outside region commutes. Hence, the (4) one which is the image of the diagram defining the compatibility

of G to $\mathfrak{L}$, $\mathfrak{L}'$ commutes, it implies that G is compatible to $\mathfrak{L}$, $\mathfrak{L}'$ since F is faithful.

The proof of Theorem 8. From the proof of Theorem 1, there exists a functor $G : \mathcal{B} \to \mathcal{A}$ and a natural isomorphism $\widetilde{G}$ such that the pair $(G, \widetilde{G})$ is a monoidal $\otimes$-functor and $\alpha : GF \cong \mathrm{id}_{\mathcal{A}}$, $\beta : FG \cong \mathrm{id}_{\mathcal{B}}$ are natural $\otimes$-isomorphisms. Moreover, we also have similar result to the product $\oplus$, in concrete there exists natural isomorphism $\breve{G}$ such that the pair $(G, \breve{G})$ is a $\oplus$-functor and α, β are natural $\oplus$-isomorphisms. Applying Lemma 10, the triplet $(G, \breve{G}, \widetilde{G})$ is an Ann-functor. Therefore, the triplet $(F, \breve{F}, \widetilde{F})$ is an Ann-equivalence. $\square$

$$\begin{tikzcd}
 & & \mathrm{FGX\,(FGY\oplus FGZ)} \arrow[ddddd, bend left, "\mathfrak{L}"] \\
 & \mathrm{FGX\ FG(Y\oplus Z)} \arrow[ur, "\mathrm{FGX}\otimes F\breve{G}"] \arrow[r, "FGX\otimes F\breve{G}"] \arrow[r, phantom, shift left=6, "(1)"] & \mathrm{FGX\ F(GY\oplus GZ)} \arrow[u, "\mathrm{FGX}\otimes\breve{F}"'] \\
 \mathrm{FG(X(Y\oplus Z))} \arrow[ur, "\widetilde{FG}"] \arrow[r, "\mathrm{F}(\widetilde{G})"] \arrow[d, "\mathrm{FG}(\mathfrak{L})"'] & \mathrm{F(GXG(Y\oplus Z))} \arrow[u, "\widetilde{F}"'] \arrow[u, phantom, shift right=6, "(2)"] \arrow[r, "\mathrm{F}(GX\otimes\breve{G})"] \arrow[ur, phantom, "(3)"] \arrow[d, phantom, "(4)"] & \mathrm{F(GX(GY\oplus GZ))} \arrow[u, "\widetilde{F}"] \arrow[d, "\mathrm{F}(\mathfrak{L}')"'] \arrow[d, phantom, shift left=8, "(8)"] \\
 \mathrm{FG(XY\oplus XZ)} \arrow[r, "\mathrm{F}(\breve{G})"] \arrow[dr, "\breve{F}G"'] & \mathrm{F(G(XY)\oplus G(XZ))} \arrow[r, "\mathrm{F}(\widetilde{G}\oplus\widetilde{G})"] \arrow[d, "\breve{F}"] \arrow[d, phantom, shift right=6, "(5)"] \arrow[dr, phantom, "(6)"] & \mathrm{F(GXGY\oplus GXGZ)} \arrow[d, "\breve{F}"'] \\
 & \mathrm{FG(XY)\oplus FG(XZ)} \arrow[r, "\mathrm{F}\widetilde{G}\oplus F\widetilde{G}"] \arrow[dr, "\widetilde{FG}\oplus\widetilde{FG}"'] \arrow[dr, phantom, shift left=4, "(7)"] & \mathrm{F(GXGY)\oplus F(GXGZ)} \\
 & & \mathrm{(FGX\ FGY)\oplus(FGX\ FGZ)} \arrow[u, "\widetilde{F}\oplus\widetilde{F}"']
\end{tikzcd}$$

$\square$

Now for any given Ann-category $\mathcal{A}$, we construct an almost strict one $\mu(\mathcal{A})$ which is Ann-equivalent to $\mathcal{A}$.

First we assume that Ann-category $\mathcal{A}$ is a strict monoidal category with operation $\oplus$ (since any Ann-category is Ann-equivalent to an one of that kind).

Definition. *The triplet $(F, \widehat{F}, \overline{F})$ is called a μ-functor if the pair $(F, \hat{F})$ is a symetric monoidal endo-equivalence to the operation $\oplus$ and the pair $(F, \overline{F})$ is a M-functor to the operation $\otimes$ satisfying following conditions:*

(i) Family $(\overline{F}_{X,Y})_Y$ is a natural $\oplus$-transfomation from $F \circ L^X$ to L^{FX},

(ii) Family $(\overline{F}_{X,Y})_X$ is a natural $\otimes$-transfomation from $F \circ R^Y$ to $L^Y \circ F$.

A μ-transformation from $(F, \hat{F}, \overline{F})$ to $(G, \hat{G}, \overline{G})$ is a natural $\oplus$-transformation $\phi : F \to G$ making the following diagram

$$\begin{array}{ccc} F(XY) & \longrightarrow & (FX)Y \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu\otimes Y} \\ G(XY) & \longrightarrow & (GX)Y \end{array}$$

commute.

Example. *For any object $A \in \mathcal{A}$, the pair $(L^A, \widehat{L}^A)$ where $\widehat{L}^A_{X,Y} = a_{A,X,Y}$ is a μ-functor. Besides, for any morphism $u : A \to B$, the natural transformation $\phi : (L^A, \widehat{L}^A, \overline{L}^A) \to (L^B, \widehat{L}^B, \overline{L}^B)$ defined by $\phi_X = u \otimes X$ is a natural μ-transformation.*

Hereafter, by $\mu(A)$ we denote the sub-category whose objects are μ-functors and whose morphisms are natural μ-transformations of the category $M(\mathcal{A})$.

We can verify following propositions.

Proposition 11. *$\mu(A)$ is a tensor category in which its tensor product $\oplus$ defined by*

$$\begin{aligned} (F \oplus G)X &= FX \oplus GX, \\ (\widehat{F \oplus G})_{X,Y} &= \nu(\hat{F}_{X,Y} \oplus \hat{G}_{X,Y}), \\ (\phi \oplus \psi)_X &= \phi_X \oplus \psi_X, \end{aligned}$$

where $\nu = \nu_{A,B,C,D} : (A \oplus B) \oplus (C \oplus D) \to (A \oplus C) \oplus (B \oplus D)$ is the morphism built uniquely from constraints a^+, c and identities in Pic-category $(\mathcal{A}, \oplus)$.

Proposition 12. *$\mu(A)$ is a Pic-category whose the associativity and unity constraints are strict. Moreover, it has*

(i) The zero-object $0^* = (\theta, \widehat{\theta}, \overline{\theta})$ given by: $\theta(X) = 0, \theta(f) = \mathrm{id}_0, \widehat{\theta}_{X,Y} = \mathrm{id}_0, \widetilde{\theta}_{X,Y} = (\widehat{\mathfrak{R}}^Y)^{-1}$,

(ii) The associativity constraint $c^*_{F,G}(X) = c_{FX,GX}$.

Proposition 13. *$\mu(A)$ is an Ann-category whose the distributivity constraints given by*

$$\mathfrak{L}^*_{F,G,H}(X) = \widehat{F}_{GX,HX},\ \mathfrak{R}^* = \mathrm{id}\,.$$

We are now ready to prove the Theorem 7 that is the main content of this section.

The proof of Theorem 7. We suppose that Ann-category $\mathcal{A}$ is a strict monoidal one to the tensor product $\oplus$. Next we show that $\mathcal{A}$ and the almost strict Ann-category $\mu(A)$ are Ann-equivalent. Defining an Ann-functor $\Phi : \mathcal{A} \to \mu(\mathcal{A})$ by

$$\begin{aligned} &\Phi(A) = (L^A, \widehat{L}^A, \overline{L}^A), \\ &\Phi(u) = L(u) : L^A \to L^B, L(u)_X = u \otimes X, \\ &\widehat{\Phi}_{A,B}(X) = \mathfrak{R}_{A,B,X}, \widetilde{\Phi}_{A,B}(X) = (a_{A,B,X})^{-1}. \end{aligned}$$

The above-defined functor Φ is an equivalence of categories. Indeed, the functor $J : \mu(A) \to \mathcal{A}$ defined by

$$J(F, \hat{F}, \widetilde{F}) = F(I),\ J(F \xrightarrow{\phi} G) = (FI \xrightarrow{\phi_I} GI)$$

is the inverse equivalence to Φ via natural isomorphisms

$$\alpha : J\Phi \cong \mathrm{id}_{\mathcal{A}},\ \beta : \Phi J \cong \mathrm{id}_{\mathcal{B}},$$

where $\alpha = r, \beta_F = \rho_F$ (ρ_F is mentioned in the proof of theorem 1). It is possible to check that ρ_F is a natural $\oplus$-transformation then it is a μ one. Applying theorem 8 we obtain that Φ is an Ann-equivalence. $\square$

Theorem 14. *The condition $c_{X,X} = id$ for any object $X \in \mathcal{A}$ (the regular condition) is neccessary and sufficient for the Ann-category $\mathcal{A}$ to be Ann-equivalent to one whose the associativity constraint is the identity.*

Proof. Assume that $\mathcal{A}$ satisfies the regular condition for the commutativity constraint, in the sense $c_{X,X} = \mathrm{id}$ for any object $X \in \mathcal{A}$. Then $\mathcal{A}$ is Ann-equivalent to $\mathcal{A}'$ which is a symetric monoidal to the tensor product $\oplus$. From the proposition 14, the commutativity constraint c^* of $\mu(\mathcal{A}')$ is the identity.
Inversely, from the commutative diagram

$$\begin{tikzcd}
\Phi(X \oplus X) \arrow[r] \arrow[d, "\Phi(c)"'] & \Phi(X) \oplus \Phi(X) \arrow[d, "c=\mathrm{id}"] \\
\Phi(X \oplus X) \arrow[r] & \Phi(X) \oplus \Phi(X)
\end{tikzcd}$$

we have $\Phi(c_{X,X}) = \mathrm{id}$ where Φ is an Ann-equivalence. Therefore $c_{X,X} = \mathrm{id}$. $\square$

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